NONNEGATIVE RICCI CURVATURE AND VIRTUALLY ABELIAN STRUCTURE

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Abstract. This short note is about fundamental groups of closed manifolds of zero sectional curvature or non-negative Ricci curvature. It includes Buser's proof on classical Bieberbach's theorem and Cheeger-Gromoll's proof on virtually abelian structure. It also offers a viewpoint of virtual abelianness from virtual nilpotency.

1. Bieberbach's theorem

Let M be a compact flat n-manifold. The universal cover of M is the standard Euclidean space $\mathbb{R}^n$. Hence to study the structure of $\pi_1(M)$, it is equivalent to study the isometry group of $\mathbb{R}^n$.

Classical Bieberbach's theorem:

Theorem 1.1. *Let M be a compact flat n-manifold. Then $\pi_1(M)$-action on $\widetilde{M} = \mathbb{R}^n$ contains n linearly independent translations. Consequently, $\pi_1(M)$ has a normal subgroup $\mathbb{Z}^n$ of finite index.*

In fact, there is a second part of Bieberbach's theorem, which states that there are only finitely many isomorphism classes of $\pi_1(M)$ in each dimension. This gives a universal bound $C(n)$ on the index.

The following algebraic lemma is helpful to get normal subgroups of finite index.

Lemma 1.2. *Let G be a group and H be a subgroup of finite index N, then there is a subgroup H' of H such that H' is normal in G and $[G : H'] \leq N^N$.*

Proof. Let $H' = \bigcap_{g\in G} g^{-1}Hg$. □

We follow [1] to prove Theorem 1.1. The idea has origins in Gromov's work on almost flat manifolds [5].

1.1. **Rotations and translations.** Any isometry α of $\mathbb{R}^n$ can be expressed as

$$\alpha = (A, a) : v \mapsto Av + a,$$

where $A \in O(n)$ is the rotational part and $a \in \mathbb{R}^n$ is the translation part.

Definition 1.3. For $A \in O(n)$, we define

$$m(A) = ||A - I_n|| = \max_{v\in\mathbb{R}^n-\{0\}} |Av - v|/|v|.$$

This measures the difference between A and the identity matrix I_n.

Lemma 1.4. *Let $A, B \in O(n)$. Then*

$$m([A, B]) \leq 2m(A)m(B).$$

Proof. It is direct to check that

$$[A,B]-I = ABA^{-1}B^{-1}-I = [A-I,B-I]A^{-1}B^{-1}.$$

Thus

$$\begin{aligned}|([A,B]-I)v| &= |[A-I,B-I]A^{-1}B^{-1}v| \\ &= |[A-I,B-I]v| \\ &\leq m(A)m(B)|v| + m(B)m(A)|v|.\end{aligned}$$

for all $v \in \mathbb{R}^n$. $\square$

Lemma 1.4 implies that if $m(A) \leq 1/2$, then $m([A,B]) \leq m(B)$.

Lemma 1.5. *Put*

$$E_A = \{v \in \mathbb{R}^n | |Av - v| = m(A)|v|\}.$$

E_A is an A-invariant subspace.

Proof. It is direct to check that E_A is invariant under A and scalar multiplication.

Let $v, w \in E_A$. Note that by parallelogram law,

$$\begin{aligned}2m(A)^2(|v|^2+|w|^2) &= 2(|Av-v|^2+|Aw-w|^2) \\ &= |A(v+w)-(v+w)|^2 + |A(v-w)-(v-w)|^2 \\ &\leq m(A)^2(|v+w|^2+|v-w|^2) \\ &= 2m(A)^2(|v|^2+|w|^2).\end{aligned}$$

It follows that $v \pm w \in E_A$. $\square$

We can decompose $\mathbb{R}^n$ as $E_A + E_A^{\perp}$, where $E_A^{\perp}$ is the orthogonal complement of E_A, which is A-invariant as well. For $A|_{E_A^{\perp}}$, we similarly define

$$m^{\perp}(A) = ||A - I_n||_{E_A^{\perp}}.$$

It is clear that if $A \neq I$, then $m^{\perp}(A) < m(A)$.

1.2. **Almost/pure translations.** We write $\Gamma = \pi_1(M)$. Recall that Γ acts cocompactly on $\mathbb{R}^n$. Thus there is $D > 0$, the diameter of M, such that for any $v \in \mathbb{R}^n$, there is $\alpha = (A, a) \in \Gamma$ such that

$$D \geq |\alpha \cdot 0 - v| = |a - v|.$$

To obtains pure translations in Γ, we will first look for almost translations. These elements have rotational parts very close to identity.

Lemma 1.6. *Let $u \in \mathbb{R}^n$ be a unit vector. For any $\epsilon > 0$, there is $\alpha = (A, a) \in \Gamma$ such that*

$$a \neq 0, \quad \measuredangle(u, a) \leq \epsilon, \quad m(A) \leq \epsilon.$$

Proof. For each $k \in \mathbb{N}$, we can find $\beta_k = (B_k, b_k) \in \Gamma$ such that $|b_k - ku| \leq D$. It is clear that

$$|b_k| \to \infty, \quad \measuredangle(u, b_k) \to 0.$$

Next we make use of the compactness of $O(n)$. After passing to a subsequence, B_k converges in $O(n)$. In this subsequence, we can pick a large i such that for all $j \geq i$, we have

$$m(B_jB_i^{-1}) \leq \epsilon, \quad \measuredangle(u, b_j) \leq \epsilon/2.$$

We consider the element $\beta_j\beta_i^{-1} = (B_jB_i^{-1}, B_jb_i + b_j) \in \Gamma$. As $j \to \infty$, the translation part $B_jb_i + b_j$ is dominated by b_j. Thus for a sufficiently large j,

$$\measuredangle(u, B_jb_i + b_j) \leq \epsilon$$

holds. This completes the proof. $\square$

The key observation in this proof is the lemma below, which states that any almost translation is indeed a pure translation.

Lemma 1.7. *Let $\alpha = (A, a) \in \Gamma$. If $m(A) \leq 1/2$, then $A = I$.*

Proof. Let
$$T = \{\alpha = (A, a) \in \Gamma \mid 0 < m(A) \leq 1/2\}.$$
Suppose that T is non-empty. We choose $\alpha \in T$ so that it has the smallest translation part $|a|$ among elements in T. Apply Lemma 1.6 to any unit vector of E_A, we can find $\beta = (B, b) \in \Gamma$ such that
$$b \neq 0, \quad |b^\perp| \leq |b^E|, \quad m(B) \leq \frac{1}{8}(m(A) - m^\perp(A)). \quad (\star)$$
We choose β with the minimal $|b| \neq 0$.

Let $\gamma = [\alpha, \beta] = (C, c)$. We claim that γ also satisfies $(\star)$ but has a smaller $|c|$, which is a contradiction to the choice of β.

Case 1: β is a pure translation. Then by direct calculation $\gamma = (I, (A - I)b)$. We decompose $c = (A - I)b$ as
$$c = c^E + c^\perp = (A - I)b^E + (A - I)b^\perp.$$
Then
$$|c^\perp| = |(A - I)b^\perp| \leq m^\perp(A)|b^\perp| \leq m(A)|b^E| = |(A - I)b^E| = |c^E|.$$
This shows that γ satisfies $(\star)$. Also,
$$\begin{aligned} |c|^2 &= |(A - I)b^E|^2 + |(A - I)b^\perp|^2 \\ &\leq m(A)^2|b^E|^2 + m^\perp(A)^2|b^\perp|^2 \\ &= (|b^E|^2 + |b^\perp|^2)/4 = |b|^2/4. \end{aligned}$$
Thus $|c| < |b|$, a contradiction.

Case 2: β is not a pure translation. By Lemma 1.4,
$$m(C) \leq 2m(A)m(B) \leq m(B).$$
By the choice of $\alpha = (A, a)$, we have $|a| \leq |b|$. It is direct to calculate that
$$(C, c) = ([A, B], -[A, B]b - ABA^{-1}a + Ab + a).$$
We write $c = (A - I)b + r$, where
$$r = -[A, B]b - ABA^{-1}a + a + b = -([A, B] - I)b - (A(B - I)A^{-1}a).$$
We estimate
$$|r| \leq m(C)|b| + m(B)|a| \leq 2m(B)|b| < 4m(B)|b^E| \leq \frac{1}{2}(m(A) - m^\perp(A))|b^E|.$$
Together with
$$\begin{aligned} |c^\perp| &\leq |(A - I)b^\perp| + |r| \leq m^\perp(A)|b^E| + |r|, \\ |c^E| &\geq |(A - I)b^E| - |r| = m(A)|b^E| - |r|, \end{aligned}$$
we conclude that $|c^\perp| < |c^E|$. Lastly,
$$|c| \leq |(A - I)b| + |r| \leq m(A)|b| + 2m(B)|b| < \frac{1}{2}|b| + \frac{1}{4}|b| < |b|.$$
We result in the desired contradiction and complete the proof. $\square$

Theorem 1.1 follows directly from Lemmas 1.6 and 1.7.

Remark 1.8. In the proof above, we never used the fact that $\pi_1(M)$-action is free. Therefore, as long as a discrete subgroup acts cocompactly on $\mathbb{R}^n$ by isometries, the conclusion in Theorem 1.1 holds.

We mention the generalized Bieberbach theorem by Fukaya-Yamaguchi [4]:

Theorem 1.9. *Let G be a closed subgroup of* $\mathrm{Isom}(\mathbb{R}^n)$*. Then G/G_0 is virtually abelian, where G_0 is the identity component of G.*

2. Virtual abelianness

The main references for this section are [2, 3].

2.1. Splitting theorem and its consequences.

We need Cheeger-Gromoll's splitting theorem:

Theorem 2.1. *Let M be a complete n-manifolds of* $\mathrm{Ric} \geq 0$*. If M contains a line, then M splits isometrically as $N \times \mathbb{R}$.*

Due to the splitting theorem, any open manifold M of $\mathrm{Ric} \geq 0$ splits isometrically as $N \times \mathbb{R}^k$, where N has no lines.

Proposition 2.2. *Let $M = N \times \mathbb{R}^k$ be a metric product, where N has no lines. Then isometry group* $\mathrm{Isom}(M)$ *splits as* $\mathrm{Isom}(N) \times \mathrm{Isom}(\mathbb{R}^k)$.

Proof. For any $g \in \mathrm{Isom}(N \times \mathbb{R}^k)$, we write

$$g(x, v) = (g_1(x, v), g_2(x, v)).$$

We need to show that g_1 and g_2 are independent of v and x, respectively.

Note that if $c(t) = (c_1(t), c_2(t))$ is a line in $N \times \mathbb{R}^k$, then c_1 must be a constant because N contains no lines. Let v and w be any two nonzero vectors in $\mathbb{R}^k$. Because $g_1(x, tv)$ is the same for all t, it follows that $g_1(x, tv) = g_1(x, 0)$. This implies

$$g_1(x, tv) = g_1(x, 0) = g_1(x, sw).$$

Thus g_1 only depends on x. In particular, g maps the slice $\{x\} \times \mathbb{R}^k$ to another slice $\{g_1(x)\} \times \mathbb{R}^k$.

Observe that $N \times \{v\}$ is orthogonal to $\{x\} \times \mathbb{R}^k$. Thus under the isometry g, $g(N \times \{v\}) =$ should be orthogonal to $g(\{x\} \times \mathbb{R}^k) = \{g_1(x)\} \times \mathbb{R}^k$. It follows that $g(N \times \{v\}) = N \times \{v'\}$. This shows that $g_2(x, v) = v'$ for all $x \in N$. □

Theorem 2.3. *Let M be a compact n-manifold of* $\mathrm{Ric} \geq 0$*. Then $\widetilde{M}$, the Riemannian universal cover of M, splits isometrically as $N \times \mathbb{R}^k$, where N is compact.*

Proof. By Theorem 2.1, $\widetilde{M}$ splits isometrically as $N \times \mathbb{R}^k$, where N has no lines. We show that N is compact.

Suppose that N is not compact. We fix a reference point $y = (x, 0) \in \widetilde{M}$. Let $c(t)$ be a unit speed ray in N. For $i \in \mathbb{N}$, we consider a segment of length $2i$:

$$c_i(t) = (c(t), 0), \quad t \in [0, 2i].$$

Because M is compact, for each i, there is $g_i \in \pi_1(M, x)$ such that

$$d(g_i(c_i(i)), y) \leq \mathrm{diam}(M).$$

By Proposition 2.2, the projection of $g_i \circ c_i$ to $\mathbb{R}^k$-factor is a single point. Passing to a subsequence if necessary, $g_i \circ c_i$ converges to a line in $\widetilde{M}$, which results in a line in N, a contradiction. □

2.2. **A finite cover.**

Theorem 2.4. *Let M be a compact n-manifold of* $\mathrm{Ric} \geq 0$. *Let $\widetilde{M} = N \times \mathbb{R}^k$ be the universal cover of M, where N is compact. Then*
(1) M has a finite cover diffeomorphic to $N \times \mathbb{T}^k$;
(2) $\pi_1(M)$ contains a normal subgroup $\mathbb{Z}^k$ of finite index.

Proof. (1). We write $\Gamma = \pi_1(M, x)$. We consider the natural projection maps from Proposition 2.2:

$$p_1 : \Gamma \to \mathrm{Isom}(N), \quad p_2 : \Gamma \to \mathrm{Isom}(\mathbb{R}^k).$$

Let H be the kernel of p_2. Since H is a discrete group of $\mathrm{Isom}(N)$, it is clear that H is finite. Γ/H acts isometrically on the quotient space $(N \times \mathbb{R}^k)/H = (N/H) \times \mathbb{R}^k$. We shall show that M has a finite cover diffeomorphic to $(N/H) \times \mathbb{T}^k$.

Let $\bar{p}_1$ and $\bar{p}_2$ be the corresponding projections of the isometry group of this intermediate cover:

$$\bar{p}_1 : \Gamma/H \to \mathrm{Isom}(N/H), \quad \bar{p}_2 : \Gamma/H \to \mathrm{Isom}(\mathbb{R}^k).$$

Note that $\bar{p}_2(\Gamma/H)$ acts isometrically and co-campactly on $\mathbb{R}^k$. By Theorem 1.1, $\bar{p}_2(\Gamma/H)$ contains a normal subgroup A isomorphic to $\mathbb{Z}^k$. Let $\hat{A} = \bar{p}_2^{-1}(A)$. Because $\bar{p}_2$ is injective, $\hat{A}$ is isomorphic to $\mathbb{Z}^k$ as well. Ideally, we wish to show that $\hat{A}$-action on $(N/H) \times \mathbb{R}^k$ is equivariant to A-action on $(N/H) \times \mathbb{R}^k$. If this is true, then $((N/H) \times \mathbb{R}^k)/\hat{A}$ is diffeomorphic to $(N/H) \times (\mathbb{R}^k)/A = (N/H) \times \mathbb{T}^k$. We claim that this can be done if $\bar{p}_1 : \hat{A} \to \mathrm{Isom}(N/H)$ admits an extension to $\mathbb{R}^k$.

Claim: If $\bar{p}_1 : \hat{A}(= \mathbb{Z}^k) \to \mathrm{Isom}(N/H)$ can be extended to a group homomorphism $\psi : \mathbb{R}^k \to \mathrm{Isom}(N/H)$, then

$$\begin{aligned} f : (N/H) \times \mathbb{R}^k &\to (N/H) \times \mathbb{R}^k \\ (z, v) &\mapsto (\psi(v)^{-1} z, v) \end{aligned}$$

is an equivariant diffeomorphism.

We can verify this claim directly. In fact, for any $\hat{a} \in \hat{A}$ with $\bar{p}_2(\hat{a}) = a \in A$,

$$a \cdot f(z, v) = a \cdot (\psi(v)^{-1} z, v) = (\psi(v)^{-1} z, a \cdot v);$$

$$\begin{aligned} &f(\hat{a} \cdot (z, v)) = f(\bar{p}_1(a) \cdot z, a \cdot v) = (\psi(a \cdot v)^{-1} \bar{p}_1(a) \cdot z, a \cdot v) \\ =&(\psi(v)^{-1} \psi(a)^{-1} \bar{p}_1(a) \cdot z, a \cdot v) = (\psi(v)^{-1} z, a \cdot v). \end{aligned}$$

However, such an extension is not always guaranteed. To overcome this, we shall replace $\hat{A}$ by a normal subgroup of finite index. Note that the closure $\overline{p_1(\hat{A})} =: G$ is a compact abelian subgroup of $\mathrm{Isom}(N/H)$. Thus its identity component G_0 is a normal toral subgroup with finite index in G. Let $\hat{B}$ the pre-image of G_0 in $\hat{A}$. $\hat{B}$ has finite index in $\hat{A}$, thus $\hat{B}$ is isomorphic to $\mathbb{Z}^k$ as well. Then

$$\bar{p}_1 : \hat{B}(= \mathbb{Z}^k) \to \bar{p}_1(\hat{B}) \subset G_0$$

can be extended to a group homomorphism

$$\psi : \mathbb{R}^k \to G_0 \subset \mathrm{Isom}(N/H)$$

by defining it through one-parameter subgroups.

Let $B = \bar{p}_2(\hat{B})$. Then by the above claim, $\hat{B}$-action on $(N/H) \times \mathbb{R}^k$ is equivariant to B-action on $(N/H) \times \mathbb{R}^k$. It follows that $((N/H) \times \mathbb{R}^k)/\hat{B}$ is diffeomorphic to $(N/H) \times \mathbb{T}^k$. It is clear that $N \times \mathbb{T}^k$ covers $(N/H) \times \mathbb{T}^k$. This proves (1).

(2) follows directly from (1). □

It is open whether there is a universal bound on the index of the abelian subgroup.

Conjecture 2.5 (Fukaya-Yamaguchi). *Given n, there is a constant $C(n)$ such that for any compact n-manifold of $\mathrm{Ric} \geq 0$, $\pi_1(M)$ has a normal abelian subgroup of index $\leq C(n)$.*

3. A viewpoint from virtual nilpotency

Milnor [7]:

Theorem 3.1. *Let M be a complete n-manifold of $\mathrm{Ric} \geq 0$. Then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leq n$.*

Gromov [6]:

Theorem 3.2. *Any finitely generated group of polynomial growth is virtually nilpotent.*

If we apply Theorems 3.1 and 3.2 above, then to prove the virtually abelian structure, we can assume that $\pi_1(M)$ is nilpotent without lose of generality.

The following theorem is important in extracting an abelian group from a nilpotent one.

Theorem 3.3. *Any compact connected nilpotent Lie group is a torus.*

Lemma 3.4. *Let G be a nilpotent subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. Let (A, x) and (B, y) be two elements of G. Then (A, x) and (B, y) commutes if and only if A and B commutes.*

Proof. The proof is linear algebra. By direct calculation, we have

$$[(A, x), (B, y)] = ([A, B], -[A, B]y - ABA^{-1}x + Ay + x).$$

Clearly if (A, x) and (B, y) commutes, so does A and B.

Conversely, if A and B commutes, then

$$[(A, x), (B, y)] = (I, -y - Bx + Ay + x),$$

which is a translation. We denote this vector as $w = -y - Bx + Ay + x$.

Since G is nilpotent, after l times of commutator calculation, we result in

$$[(A, x), [..., [(A, x), (I, w)]]] = (I, 0).$$

It is easy to verify that the left hand side equals to $(I, (A - I)^l w)$. Thus

$$(A - I)^l w = 0$$

for some l. With the fact that $A \in O(n)$, we have

$$(A - I)^l w = 0 \text{ if and only if } (A - I)w = 0.$$

Therefore, $Aw = w$. Similarly, we have $Bw = w$. Since A and B commutes, they share the same eigen-space decomposition. We define a subspace E as

$$E = \{v \in \mathbb{R}^n \mid Av = v = Bv\}$$

and decompose $\mathbb{R}^n$ as $E + E^\perp$, where $E^\perp$ is the orthogonal complement of E. We write $x = x^E + x^\perp$ and $y = y^E + y^\perp$ according to this decomposition. Then

$$w = Ay^\perp - y^\perp + x^\perp - Bx^\perp,$$

which is in $E^\perp$. Since $w \in E$, we conclude that $w = 0$ and complete the proof. $\square$

Proposition 3.5. *Let Γ be a discrete nilpotent subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. Then Γ is virtually abelian.*

Proof. We consider the group homomorphism

$$\pi : \mathrm{Isom}(\mathbb{R}^n) \to O(n)$$
$$(A, x) \mapsto A$$

Let H be the closure of $\pi(\Gamma)$ in $O(n)$. H is a compact nilpotent Lie group (could be a finite group). Let H_0 be the identity component of H, which has finite index in H. By Theorem 3.3, H_0 is a torus. Therefore, by Lemma 3.4, $\pi^{-1}(H_0)$ is an abelian subgroup of finite index in Γ. □

Proposition 3.6. *Let Γ be a discrete nilpotent subgroup of $K \times \mathrm{Isom}(\mathbb{R}^k)$, where K is a compact Lie group. Then Γ is virtually abelian.*

Proof. Let

$$p_1 : \Gamma \to K, \quad p_2 : \Gamma \to \mathrm{Isom}(\mathbb{R}^k)$$

be the natural projections. By Proposition 3.5, $p_2(\Gamma)$ has an abelian subgroup A_2 of finite index. Also, by the same argument in Proposition 3.5, $p_1(\Gamma)$ has an abelian subgroup A_1 of finite index. Clearly, $\Gamma \cap (A_1 \times A_2)$ is the desired abelian subgroup of finite index in Γ. □

Remark 3.7. In the proof of Proposition 3.5, H_0 is indeed central in H. Using this, one can improve the result in Propositions 3.5 and 3.6: the center of Γ has finite index in Γ.

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